J. Fluid Mech. (1999), *vol.* 391, *pp.* 373–376. Printed in the United Kingdom

A note on constraints in turbulence modelling

By PHILIPPE R. SPALART[1] AND CHARLES G. SPEZIALE[2]†

[1]Boeing Commercial Airplanes, PO Box 3707, Seattle, WA 98124, USA

[2]Boston University, Boston, MA 02215, USA

(Received 26 May 1998 and in revised form 23 March 1999)

We show that the class of constitutive relations for turbulence models put forward by Wang (1997) in this journal conflicts with dimensional analysis, unless the turbulent Reynolds stresses were to be tied to the molecular viscous stresses everywhere in the flow. We then reiterate, using counter-examples, that the controversial postulate of material frame-indifference is unfounded for turbulence, and is counter-productive in the quest for accuracy. We add a comment on the role of acceleration.

1. Overview

Wang (1997) which we call W97 from here on, presents theoretical developments regarding turbulence models and describes them as 'in sharp contrast to the literature' in places, but 'systematic and rigorous'. We have not detected any algebra errors. Therefore, the *assumptions* of the work are under scrutiny. They lead to strong unstated consequences. We first challenge the exclusion of some parameters using dimensional arguments, which suffer no controversy, and the textbook view of the role of viscosity within turbulence. We then address the independent issue of material frame-indifference (MFI) in turbulence, which has long been controversial and again differ with W97, because of the known effects of rotation on turbulence and of the stress-tensor anisotropy.

2. Dimensional considerations

The body of W97 begins by specifying a class of constitutive relations giving the Reynolds-stress tensor $\boldsymbol{R}$. This apparently abrupt step is a normal one, since modelling amounts to proposing a description of turbulence that is considerably simplified and numerically manageable, yet useful in some settings. In its equation (1), W97 restricts the constitutive relation to use only the mean velocity vector v, the mean velocity-gradient tensor $\boldsymbol{L}$, and 'Other Parameters', OP which are local, such as thermodynamic state variables. Thus, $\boldsymbol{R} = \boldsymbol{F}(\mathrm{OP}, v, \boldsymbol{L})$. This constitutive relation may be linear or nonlinear; it does not modify the arguments of dimensional analysis.

W97 then eliminates v as a valid variable, based on invariance to Galilean transformations. We agree on this point. This leaves only $\boldsymbol{L}$ and OP, equation (7) in W97. The very unusual exclusion of parameters such as the turbulent kinetic energy, k, is emphasized in the paper. The kinematic Reynolds stresses $\boldsymbol{R}_{ij}/\rho = \overline{u_i' u_j'}$ have the

† Professor Speziale became gravely ill while this paper was being revised and passed away on 30 April 1999.

dimension of a velocity squared. Now if we consider an incompressible flow, no combination of $\boldsymbol{L}$ and OP has that dimension, unless the molecular viscosity is used (OP include density and pressure, but the pressure in an incompressible flow contains an arbitrary additive constant, so that p/ρ cannot be used).

An accepted concept of turbulence has long been that its fluctuations induce much more powerful mixing and momentum transfer than the molecular phenomena can (Tennekes & Lumley 1972, p. 3; Townsend 1976, p. 1). As a rule, the 'turbulence Reynolds number' is large, and it varies by orders of magnitude. An essential example of our understanding and modelling of turbulence is the defect law in channels flows, pipe flows, and boundary layers (Tennekes & Lumley 1972, p. 153); molecular viscosity does not appear in it. The same applies to several of Kolmogorov's essential results (Tennekes & Lumley 1972, pp. 262–267). The role of viscosity is weak, with exceptions such as the viscous sublayer very near a solid wall. We find that there is no dimensionally correct high-Reynolds-number turbulence model in the class assumed by W97.

3. Material frame-indifference

3.1. *Translation and rotation of a closed incompressible flow*

In the long-standing controversy over the role of rotation and of acceleration in turbulence models, it is patent that inspections of the transport equations for the Reynolds stresses have failed to lead to a consensus. Of course these transport equations do not exhaust turbulence; they are local in space and time, and unclosed. Therefore, we prefer to consider an entire flow, the solution of an initial-boundary-value problem with finite spatial and temporal extent, and clear statements of its boundary conditions. Exact results (none of which are new, of course) then provide firm facts in the controversy, although only in the incompressible limit.

We consider a fluid with constant density ρ in a closed container. The motion can be created by a prescribed motion of the boundaries, for instance a fan in a wind tunnel, or by prescribed initial conditions, and mass flow, and/or a pressure jump across periodic boundaries. We refer the motion to a set of axes, which we call 'boundary axes', in which the agent that drives the flow will be unchanged. Let $\boldsymbol{X}$ be the position vector with respect to these axes, t the time, and $\boldsymbol{U}(\boldsymbol{X}, t)$ the velocity vector with respect to the boundary axes: $\boldsymbol{U} \equiv \mathrm{d}\boldsymbol{X}/\mathrm{d}t$.

A similar thought experiment consists in obtaining the same mean motion $\boldsymbol{U}$ relative to an inertial and a non-inertial frame, by applying a suitable distribution of body forces. The validity of MFI is then tested by analysing the Reynolds stresses. Such tests have become very clear since rotating homogeneous shear flow and initially isotropic turbulence have been examined by direct numerical simulations (Speziale 1991). These showed definitively that different values of the Reynolds stresses were obtained depending on the state of rotation of the system.

We now consider imposing a solid-body motion (SBM) on the boundary of the fluid domain; this means that the boundary conditions on $\boldsymbol{U}$ are the same with or without the SBM. That motion can be described by a translation at a velocity vector $\boldsymbol{u}(t)$, and a rotation with angular velocity vector $\Omega(t)$. The continuity condition is insensitive to the SBM; therefore, only the momentum equation requires attention.

The following three results hold. The solution $\boldsymbol{U}(\boldsymbol{X}, t)$ is insensitive to: (I) the translation velocity $\boldsymbol{u}$ and: (II) the acceleration $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$. Independence from a constant $\boldsymbol{u}$ is obvious based on Galilean principles. Independence from $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$ holds because adding $(\rho \mathrm{d}\boldsymbol{u}/\mathrm{d}t)\cdot \boldsymbol{X}$ to the pressure will restore balance in the momentum equation, in an incompressible flow.

If the flow is two-dimensional, the solution $\boldsymbol{U}(\boldsymbol{X},t)$ is furthermore (III) insensitive to the rotation Ω, provided Ω is independent of time. This holds because the Coriolis term $\Omega \times \boldsymbol{U}$ can be offset by the pressure term again (the additional pressure being here $|\Omega|\Psi$ where Ψ is the stream function), and similarly for the centrifugal term (with $|\Omega|^2|\boldsymbol{X}|^2/2$). If $\mathrm{d}\Omega/\mathrm{d}t \neq 0$, there is an effect, of course.

We contend that the proponents of MFI in turbulence have not proven a theorem similar to (III) in three dimensions. We also contend that both experiments and simulations provide firm and relevant counter-examples to such a theorem; see for instance Watmuff, Witt & Joubert (1985), Kristoffersen & Andersson (1993), and Speziale (1991).

Result (II) is of separate interest because even recent papers introduce the acceleration or the pressure gradient in turbulence models (Girimaji 1997). Except for a non-trivial cancellation, such models will violate result (II), which is rigorous. It hardly seems reasonable to disregard the limit of an incompressible closed flow when designing a turbulence model.

3.2. *Specific results in W97*

Broadening the class of constitutive relations relative to W97, while it is contrary to that author's position, removes the dimension problem and allows us to consider the bulk of the paper's contentions, which relate to the anisotropy of the $\boldsymbol{R}$ tensor. Thus we replace equation (7) in W97 with the less restrictive $\boldsymbol{R} = \boldsymbol{F}(\mathrm{OP}, \boldsymbol{L}, \mathrm{TP})$ where TP represents Turbulence Parameters, such as a velocity and a time scale. Again we examine direct consequences of the hypotheses and counter-examples, instead of challenging the initial step based on some presumed authority or on a simple inspection of equations (result (III) above taught us that merely 'seeing' a Coriolis term in an equation is not a proof that the term matters). The consequences are not as arresting as those in § 2, in the sense that the conflict with common turbulence models is not strong. On the other hand, in W97 the MFI results are presented as rigorous properties, not just as working assumptions. Therefore, any conflict is substantial.

The first result that stands out is Theorem 4, which states that only the strain-rate tensor $\boldsymbol{D}$, the symmetric part of $\boldsymbol{L}$, is a valid entry in the constitutive relation that provides $\boldsymbol{R}$. The anti-symmetric part $\boldsymbol{W}$, which represents rotation, is excluded. Now $\boldsymbol{W}$ may or may not enter the equations that determine TP (W97 left that question unanswered, having excluded TP). A model in which $\boldsymbol{W}$ does not enter in any way fails to recognize the SBM discussed in § 3.1 (a solution $(\boldsymbol{U}, \boldsymbol{R})$ of the model equations without SBM also satisfies the equations with it, because both $\boldsymbol{D}$ and the molecular viscosity are insensitive to the SBM). Models in this class are having long and useful careers. However, it is widely accepted that failing to respond to a system rotation is a limitation for a model; in addition, intuition suggests that the local rotation represented by $\boldsymbol{W}$ is a physically attractive variable for an empirical turbulence model. A dependence on $\boldsymbol{W}$ is both appropriate and promising in terms of accuracy. This dependence is present in second-order closures which do not use an $\boldsymbol{F}$-type relation and perform rather well on homogeneous shear flow in a rotating frame (Speziale 1991). Some models use only $\boldsymbol{D}$ in the constitutive relation, but make TP depend on $\boldsymbol{W}$, which allows them to reproduce gross effects of rotation (Spalart & Shur 1997).

A second result which can be tested is Theorem 5: eigenvectors of $\boldsymbol{D}$ are also eigenvectors of $\boldsymbol{R}$. This property is satisfied by the linear eddy-viscosity constitutive relation, a useful approximation. However, it is definitely violated by experimental and direct-simulation results even in flows as simple as a channel, two-dimensional in the mean with a velocity profile $\mathbf{U}(y)$. There, two eigenvectors of $\boldsymbol{D}$ are at $\pm 45°$ to the flow direction x and the gradient direction y, because the only non-zero entry in the $\boldsymbol{L}$ tensor is $\partial\mathbf{U}/\partial y$. Those of $\boldsymbol{R}$ are not at $\pm 45°$, because the streamwise Reynolds stress $\overline{u'^2}$ is consistently larger than the wall-normal stress $\overline{v'^2}$, Townsend (1976), p. 290. This deviation has an influence on the mean flow as soon as it has curvature or pressure gradients. Therefore, a model that satisfies Theorem 5 is limited. We are not asserting that the use of $\boldsymbol{W}$ brings a vast improvement, but we are asserting that there is no rigorous argument against it.

These limitations are meaningful, because there is no indication in W97 that the models considered are restricted to a narrow class of flows. The paper repeatedly refers to models such as k–ϵ, which are not viewed as narrow and are constantly applied in wall turbulence. W97 contains the statement that 'the results are limited to nearly homogeneous flows with negligible effect of memory', but it seems clear that its extension to a wider class of flows would also include MFI.

We conclude, at odds with W97 and other papers before it (Speziale 1979), that MFI rapidly leads to conflicts with accepted facts, is not a property of turbulence, and is not a proper constraint to place on turbulence models.

Dr Allmaras reviewed the manuscript.

REFERENCES

Girimaji, S. S. 1997 A Galilean invariant explicit algebraic Reynolds stress model for turbulent curved flows. *Phys. Fluids* **9**, 1067–1077.

Kristoffersen, R. & Andersson, H. I. 1993 Direct simulation of low-Reynolds-number turbulent flow in a rotating channel. *J. Fluid Mech.* **256**, 163–197.

Spalart, P. R. & Shur, M. 1997 On the sensitization of simple turbulence models to rotation and curvature. *Aerospace Sci. Technol.* **1**, 5, 297–302.

Speziale, C. G. 1979 Invariance of turbulent closure models. *Phys. Fluids* **22**, 1033–1037.

Speziale, C. G. 1991 Analytical methods for the development of Reynolds stress closures in turbulence. *Ann. Rev. Fluid Mech.* **23**, 107–157.

Tennekes, H. & Lumley, J. L. 1972 *A First Course in Turbulence*. MIT Press.

Townsend, A. A. 1976 *The Structure of Turbulent Shear Flow*. Cambridge University Press.

Wang, L. 1997 Frame-indifferent and positive-definite Reynolds stress-strain relation. *J. Fluid Mech.* **352**, 341–358 (referred to herein as W97).

Watmuff, J. H., Witt, H. T., & Joubert P. N. 1985 Developing turbulent boundary layers with system rotation. *J. Fluid Mech.* **157**, 405–448.